Introduction to Mathematics and Modeling

lecture 5

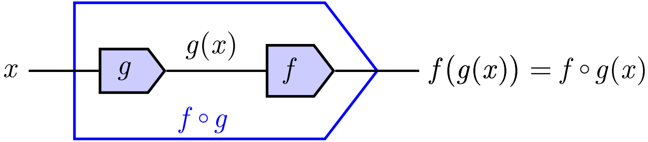
The chain rule and optimization

UNIVERSITY OF TWENTE.

academic year : 18-19
lecture : 5
build : December 10, 2018
slides : 31

Part 2

1. Section 3.6: the chain rule
2. Section 3.8: derivatives of logarithms (only pages 176–181)
3. Section 4.1: extreme values



- The **composition of f and g** is the function that maps x to $f(g(x))$
- The composition is denoted as $f \circ g$, and is pronounced as "f after g".
- Example: let $f(x) = x^2$ and let $g(x) = x + 1$, then

$$f \circ g(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$

and

$$g \circ f(x) = g(f(x)) = g(x^2) = x^2 + 1.$$

⚠ In general $f \circ g$ and $g \circ f$ are *not* identical.

- Let $f(x) = ax + b$ and $g(x) = \sin(x)$ and define $h = f \circ g$, then

$$h(x) = f \circ g(x) = f\big(g(x)\big) = a\sin(x) + b$$

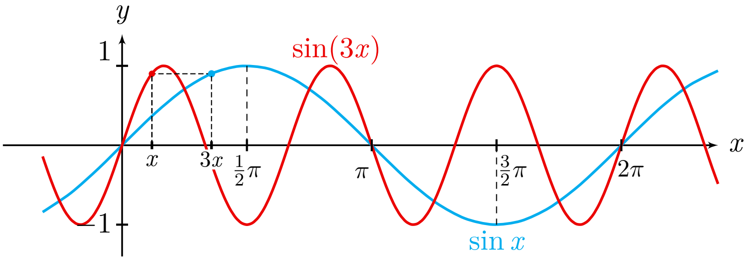
- Using the sum rule and constant multiple rule we know that

$$h'(x) = a\cos(x)$$

- Now let $h = g \circ f$ then

$$h(x) = g\big(f(x)\big) = \sin(ax + b)$$

The sum- and constant multiple rule cannot be applied



- Consider the special case $h(x) = \sin(3x)$. The graph of h is obtained by scaling $\sin x$ in horizontal direction.
- The slopes of all tangents are scaled too!
- By scaling back $\sin(3x)$ in vertical direction, this effect is cancelled out:

$$\frac{d}{dx}\left(\tfrac{1}{3}\sin(3x)\right) = \cos(3x),$$

in other words: $\dfrac{d}{dx}\sin(3x) = 3\cos(3x)$.

- We see that

$$f(x) = \sin(ax) \quad \Rightarrow \quad f'(x) = a\cos(ax)$$

- By shifting a graph horizontally, the slopes must shift accordingly:

$$f(x) = \sin(ax+b) \quad \Rightarrow \quad f'(x) = a\cos(ax+b)$$

Chain rule, simple version

Let f be a differentiable function. Then for any constant a and b the following holds:

$$\frac{d}{dx}(f(ax+b)) = af'(ax+b).$$

⚠ Warning: $\frac{d}{dx}(f(ax+b))$ is the derivative of the composition $f(ax+b)$, but $f'(ax+b)$ is the composition of f' and $y = ax+b$.

- The derivative of $\sin(2x)$ is $2\cos(2x)$.

- Define $y = \sqrt{5-3x}$, then

$$\frac{d\,y}{d\,x} = -\frac{3}{2\sqrt{5-3x}}$$

since $\frac{d}{d\,x}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

Also: write $5 - 3x = (-3)x + 5$, hence $a = -3$ and $b = 5$.

- $$\frac{d}{d\,x}\left(\frac{1}{2\,e^{x}}\right) =$$

- See lecture 4: if we define $f(x) = a^x$, then

  $$f'(x) = k_a\, a^x$$

  where

  $$k_a = \lim_{h \to 0} \frac{a^h - 1}{h} = f'(0).$$

- With the simple version of the chain rule we can prove: $\boxed{k_a = \ln a}$

  $$\frac{d}{d\,x}\,(a^x) =$$

Chain rule

Let f and g be differentiable functions, then

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x).$$

- In words: multiply the composition of the derivative of f with g by the derivative of g.
- Work inward:
  - differentiate the 'outer function' f, but keep the 'inner function' g intact;
  - then multiply with the derivative of the 'inner function' g.

Example

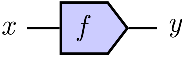
Find the derivative of $h(x) = (3x^2 + 1)^2$.

- The function h is equal to $h = f \circ g$, where

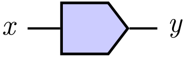
$$f(x) = x^2 \quad \text{and} \quad g(x) = 3x^2 + 1.$$

- Apply the chain rule:

$$h'(x) =$$

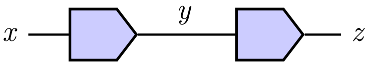


$y = f(x)$



anonymous

- If a function is named f, the derivative is denoted as f'.
- If y is an anonymous function of x, the derivative is denoted as $\frac{dy}{dx}$.



If y is a function of x and z is a function of y, then z is (by composition) a function of x. In this case the chain rule is

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}.$$

⚠ Note that $\frac{dz}{dy}$ is expressed in terms of y, hence afterwards you should replace all occurrences of y with the corresponding expression in x.

Example

Let $y = 3x^2 + 1$ *and* $z = y^2$, *find* $\dfrac{dz}{dx}$.

- Apply the chain rule (anonymous variant):

$$\frac{dz}{dx} =$$

Example

Find the derivative of $f(x) = \dfrac{1}{\sqrt{x^2+1}}$.

- Avoid using the quotient rule by writing

$$f(x) = \left(x^2+1\right)^{-1/2}.$$

- Apply the chain rule:

$$f'(x) =$$

Example

Calculate the derivative of $f(x) = \sqrt{\dfrac{1-x^2}{1+x^2}}$.

- Combine the chain rule with the quotient rule:

$$f'(x) =$$

- The **logarithmic function with base** a is the inverse of the base-a exponential function:

$$\boxed{y = a^x \quad \Longleftrightarrow \quad x = \log_a y}$$

- The **natural logarithm** is the logarithm with base e:

$$\boxed{\ln x = \log_e x} \quad \text{where} \quad \boxed{e \approx 2.71828\ldots}$$

- Examples:

$$\log_2 8 = 3 \qquad \text{because} \qquad 2^3 = 8$$

$$\log_{10} 100 = 2 \qquad \text{because} \qquad 10^2 = 100$$

$$\ln e\sqrt{e} = \tfrac{3}{2} \qquad \text{because} \qquad e^{\frac{3}{2}} = e\sqrt{e}$$

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a \frac{1}{y} = -\log_a y$$

$$\log_a x^p = p \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}, \quad \text{in particular} \quad \log_a x = \frac{\ln x}{\ln a}$$

$$a^x = b^{x \log_b a}, \quad \text{in particular} \quad a^x = e^{x \ln a}$$

- Note that e^x and $\ln(x)$ are each others inverse:

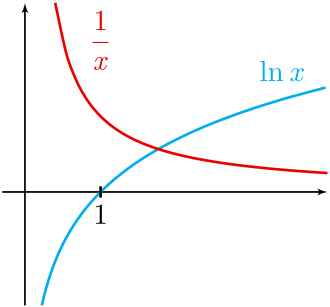
$$e^{\ln(x)} = x.$$

- Now take derivatives on both sides and apply the chain rule to the left-hand side:

$$e^{\ln(x)}\ln'(x) = 1,$$

$$x\ln'(x) = 1,$$

$$\ln'(x) = \frac{1}{x}.$$



⚠ This holds for $x > 0$.

Theorem

The derivative of $\log_a x$ *is* $\dfrac{1}{x\ln(a)}$.

- From the change-of-base formula for logarithms follows

$$\log_a(x) = \frac{\ln x}{\ln a}.$$

- Apply the constant-multiple rule:

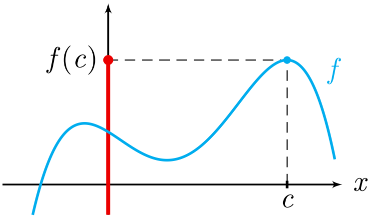
$$f'(x) =$$

Example

Find the derivative of $f(x) = \ln(x^2 + 3)$.

- Apply the chain rule:

$$f'(x) =$$



Consider a function $f\colon D \to \mathbb{R}$.

- f has an **absolute maximum** in c if

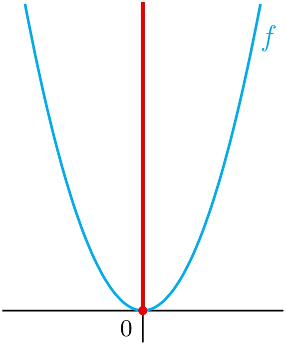
$$f(x) \leq f(c) \qquad \text{for all } x \in D$$

- f has an **absolute minimum** in c if

$$f(x) \geq f(c) \qquad \text{for all } x \in D$$

⚠ Extreme values do not necessarily have to exist!

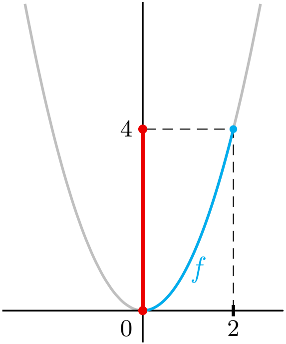
?? If they exist, how do we find them?



On $D = (-\infty, \infty)$ the function $f(x) = x^2$ has

- an absolute minimum in $x = 0$;
- no absolute maximum.

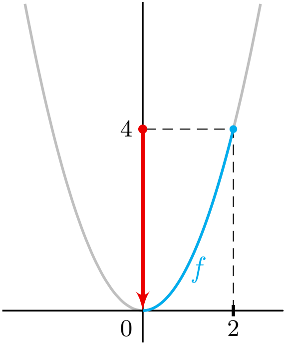
The range of f is $[0, \infty)$.



On $D = [0, 2]$ the function $f(x) = x^2$ has

- an absolute minimum in $x = 0$;
- an absolute maximum in $x = 2$.

The range of f is $[0, 4]$.



On $D = (0, 2]$ the function $f(x) = x^2$ has

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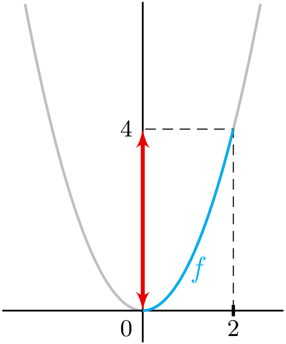
The range of f is $(0, 4]$.



On $D = (0, 2)$ the function $f(x) = x^2$ has

- no absolute minimum;
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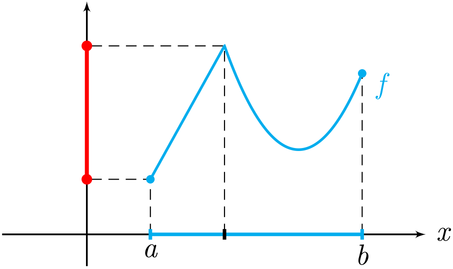
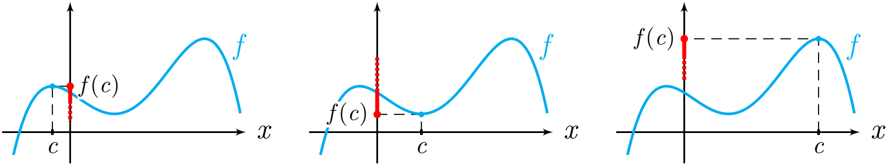


Extreme Value Theorem Theorem 1, page 223

*A **continuous function** on a **finite closed interval** attains both an absolute maximum and an absolute minimum.*

⚠ The theorem tells us that extreme values do exist, but *not* where to find them!



Definition

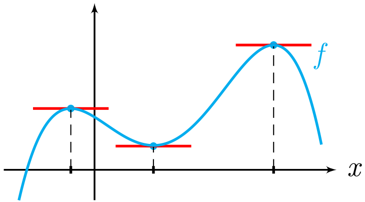
Consider a function $f\colon D \to \mathbb{R}$.

- *f has a **local maximum** in c if*

  $$f(x) \leq f(c) \qquad \text{for all } x \text{ in an environment of } c$$

- *f has an **local minimum** in c if*

  $$f(x) \geq f(c) \qquad \text{for all } x \text{ in an environment of } c$$



First Derivative Theorem Theorem 2, page 225

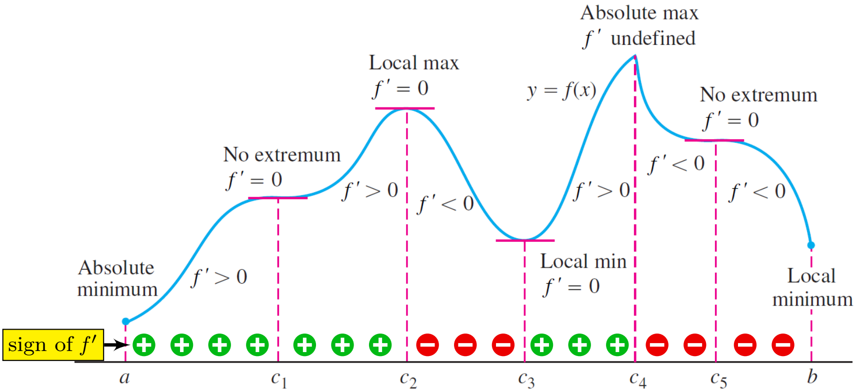
If f is differentiable at c and f attains a local maximum or local minimum at c then $f'(c) = 0$.

Definition

The number c is a **critical point of f** if

$$f'(c) = 0 \quad \text{or} \quad f \text{ is not differentiable at } c.$$

Critical points are *candidates* for a local maximum or minimum.



The function F is defined on the domain $[a, b]$.

- Critical points: c_1, c_2, c_3, c_4 and c_5.
- Local extremes: a, c_2, c_3, c_4 and b.
- Absolute extremes: a and c_4.

Recipe for computing the extreme values of a continuous function

$f\colon [a, b] \to \mathbb{R}$

1. Find *all* critical points of f in $[a, b]$, i.e., solve the equation $f'(x) = 0$ and retain all solutions x in $[a, b]$; then add all points where f is not differentiable.
2. Evaluate f at the critical points and at the end points $x = a$ and $x = b$.
3. Take the largest and smallest values found in step 2: these are the absolute maximum and minimum of f on the interval $[a, b]$.

Example

Find extremes for $f(x) = 10x(2 - \ln x)$ *on* $[1, e^2]$.

1. Find the critical points:

2. Evaluate f at the critical points and at the endpoints:

$f(1) =$ $\qquad\qquad$ $f(e^2) =$

3. Take the largest and smallest values of step 2:
   - The absolute maximum is ☐
   - The absolute minimum is ☐

Example

Find extremes for $f(x) = xe^{-x}$ *on* $[-1, 1]$.

1. Find the critical points:

2. Evaluate f at the critical points and at the endpoints:

   $f(-1) =$ $f(1) =$

3. Take the largest and smallest values of step 2:
   - The absolute maximum is ☐
   - The absolute minimum is ☐

Example

Find extremes for $f(x) = 3x^2 - 2x^3$ *on* $\left[-\frac{1}{2}, 2\right]$.

1. Find the critical points:

2. Evaluate f at the critical points and at the endpoints:

   $f\left(-\frac{1}{2}\right) =$ $\qquad$ $f(2) =$

3. Take the largest and smallest values of step 2:

   - The absolute maximum is ☐
   - The absolute minimum is ☐